Prime Element Stability in Ring Extensions

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- Prime ideals have been well studied in ring extensions.
- Recently, papers such as [5] by de Castro and [12] by Dutta have considered prime elements.
- We will be looking at a conjecture proposed by de Castro in [5] and related problems.

Rings

Definition

A ring, $(R, +, \cdot)$ is a nonempty set R with two binary operations $+$ and $\cdot$ satisfying:

1. $(R, +)$ is an abelian group
2. $\cdot$ is associative
3. for all $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

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Example

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}[x]$ are all commutative rings with identity.

Zero divisors and Integral Domains

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- A domain implies the cancellative property.
- For our research, we will be assuming all rings are domains.

Units, Irreducibles, and Primes

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We call an element $a \in R$ a unit if there exists $b \in R$ such that $ab = 1_R$. Futhermore, we typically write $U(R)$ to denote the group of units in a ring R.

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- Our research revolves around analyzing these prime elements.

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Example

Consider $\mathbb{Z}$.

- The ideals are in the form (n) for all $n \in \mathbb{Z}$.
- The prime ideals are (0) and (p) for all prime p.
- The maximal ideals are in the form (p) for all prime p.

Ideals (continued)

Theorem

Let R be a commutative ring with 1 and $I \subset R$ a proper ideal. Then,

1. *I is prime if and only if R/I is an integral domain.*
2. *I is maximal if and only if R/I is a field.*

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Example

Consider $\mathbb{Q}[x, y]$.

- The ideal (x) is prime since $\mathbb{Q}[x, y]/(x) \cong \mathbb{Q}[y]$, which is a domain.
- The ideal (x, y) is maximal since $\mathbb{Q}[x, y]/(x, y) \cong \mathbb{Q}$, which is a field.

Krull Dimension

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- A ring with Krull dimension 1 implies that all nonzero prime ideals are maximal.

Integral Extension

Definition

Let $R \subseteq T$ be domains. We say that $t \in T$ is integral over R if t is a root of a monic polynomial

$$p(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0 \in R[x].$$

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Example

- $\sqrt{2}$ is integral over Z.
- $\mathbb{Z} \subset \mathbb{Z}[\sqrt{2}]$ is an integral extension.

Overrings

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Example

Consider the extension $\mathbb{Z}[2i] \subseteq \mathbb{Z}[i]$. Since the quotient field of $\mathbb{Z}[2i]$ is $\mathbb{Q}[i]$ and $\mathbb{Z}[i] \subseteq \mathbb{Q}[i]$, we know $\mathbb{Z}[i]$ is a overring of $\mathbb{Z}[2i]$.

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- Consider $R \subseteq T$ such that T is an overring of R. Notice that for all $t \in T$, we can write $t = \frac{r_1}{r_2}$ for $r_1, r_2 \in R$.
- Both R and T share the same quotient field.

Conductor Ideal

Definition

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Example

Consider the extension $\mathbb{Z}[x^2, x^3] \subset \mathbb{Z}[x]$. The conductor is $I = x^2\mathbb{Z}[x]$ since $a \cdot \mathbb{Z}[x] \in \mathbb{Z}[x^2, x^3]$ for all $a \in I$. Additionally, we can show I covers all elements through casework on coefficients.

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Proposition

If $R \subseteq T$ is an integral extensions, then $\dim(R) = \dim(T)$.

Our Question

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Given an extension $R \subset T$ of domains, we study when an element $p \in R$ that is prime in R (and possibly prime in T) remains prime in every intermediate ring S with $R \subseteq S \subseteq T$, and we determine various extension types which exhibit such stability.

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Example

Consider the extensions $\mathbb{Z} \subset \mathbb{Z}[3i] \subset \mathbb{Z}[i]$, 3 is prime in both $\mathbb{Z}$ and $\mathbb{Z}[i]$. However, since $(3i)^2 = -9 = 3 \cdot (-3)$ but $3 \nmid_{\mathbb{Z}[3i]} 3i$, 3 isn't a prime in $\mathbb{Z}[3i]$. This example shows that prime stability doesn't always hold.

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Theorem (de Castro)

Let $\mathbb{Z}[\omega]$ be a quadratic integer ring, and consider $\mathbb{Z} \subset \mathbb{Z}[n\omega] \subset \mathbb{Z}[\omega]$. Let $p \in \mathbb{Z}$ be prime in both $\mathbb{Z}$ and $\mathbb{Z}[\omega]$. Then p is prime in $\mathbb{Z}[n\omega]$ if and only if $\gcd(n, p) = 1$.

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Theorem (Dutta)

If R is a Noetherian domain, T is the integral closure of R, then any prime element of R remains prime in T.

Prime Stability in 1-dimensional Integral Overring

Conjecture (de Castro)

Let R be 1-dimensional and T be an integral overring of R. If p is prime in R and T, then p is prime in every intermediate ring S.

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Example (de Castro)

Consider the extensions

$$\mathbb{Z}[2^m i] \subseteq S \subseteq \mathbb{Z}[i]$$

As 2^m is relatively prime to 3, for any $m \geq 0$, we know that 3 is also prime in $\mathbb{Z}[2^m i]$. We proceed to consider what S could be. We claim that S will always be of the form

$$S = \mathbb{Z}[2^n i]$$

for $0 \leq n \leq m$. Thus, 3 is prime in all possible intermediate extensions.
We also note that $\mathbb{Z}[i]$ is integral over $\mathbb{Z}[2^m i]$ and they are both 1-dimensional. They also have the same quotient field $\mathbb{Q}(i)$. So this is an explicit example of the above conjecture.

Prime Stability in 1-dimensional Integral Overring

Lemma

Let $R \subseteq T$ be an extension of 1-dimensional domains and $p \in R$ prime in T. Then, the following are equivalent:

1. *p is prime in R.*
2. *$pT \cap R = pR$.*

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Example

Let $R = \mathbb{Z}[pi]$ and $T = \mathbb{Z}[i]$ where $p \in \mathbb{Z}$ is a prime in T (for example any prime with $p \equiv 3 \pmod 4$). Note p is not prime in R since $pi \cdot pi = -p^2$ is a multiple of p in R but $p \nmid_R pi$. Furthermore, we see that $pi \in R$ but $i \notin R$, which means $pi \in pT \cap R$ and $pi \notin pR$. Thus, $pT \cap R \neq pR$, matches the lemma.

Prime Stability in 1-dimensional Integral Overring

Theorem (Coykendall-Kettinger-Liu-Mao-Zhao, 2025)

Let R be 1-dimensional and T be an integral overring of R. If p is prime in R and T, then p is prime in every intermediate ring S.

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Let R be 1-dimensional and T be an integral overring of R. If p is prime in R and T, then p is prime in every intermediate ring S.

- Since integral extension preserves Krull dimension, $\dim R = \dim S = \dim T = 1$.
- Apply the lemma above, it suffices to show that $pT \cap S = pS$.
- We only need to show that if $t \in T$ satisfies $pt \in S$, then $t \in S$.
- The rest of the proof is to show with some technical details that $t \in S$.

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Example

Let $F_0 = \mathbb{Q} \subset F_1 = \mathbb{Q}(\sqrt{2}) \subset F_2 = \mathbb{Q}(\sqrt[4]{2}) \subset F_3 = \mathbb{Q}(\sqrt[8]{2}) \subset \dots$ be algebraic field extensions, $L = \bigcup_{i=0}^{\infty} F_i$, consider the ring extension $R = F_0 + xF_1[x] + x^2F_2[x] + x^3F_3[x] + \cdots \subset L[x] = \bar{R}$. Note that $\bar{R}$ is a 1-dimensional integral overring of R. We can see that $x+1$ is prime in R and $\bar{R}$, so $x+1$ must be prime in all intermediate rings between R and $\bar{R}$.

Prime Stability in 1-dimensional Integral Overring

The restriction to 1-dimensional domains in the theorem is essential.

Example

Let F be a field and set

$$R := F[x, y^2, y^3] \subset S := F[x, xy, y^2, y^3] \subset T := F[x, y].$$

Here T is an integral overring of R, since y is integral over R (satisfying $t^2 - y^2 \in R[t]$), and hence both S and T are integral over R. Note also that $\dim R = \dim T = 2$, so R is not 1-dimensional. Consider $p := x \in R$. In R we have

$$R/(x) \cong F[y^2, y^3],$$

which is an integral domain, so x is prime in R. Similarly, in $T = F[x, y]$ the ideal (x) is prime, since

$$T/(x) \cong F[y]$$

is a domain. However, in the intermediate ring S we have the factorization

$$xy^3 = (xy) \cdot y^2,$$

and x divides neither factor in S. Thus x is not a prime element of S.

Prime Stability via Conductor Coprimality

Example

Let $K = \mathbb{Q}(\sqrt{5})$ and $\mathscr{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. Set

$$R = \mathbb{Z} + 6\mathscr{O}_K, \quad I = (R : \mathscr{O}_K) = 6\mathscr{O}_K,$$

and for each divisor $d \mid 6$ define

$$S_d = \mathbb{Z} + d\mathscr{O}_K.$$

Then the only intermediate rings are S_2 and S_3. Let $p = 7$. Since $\gcd(7, 6) = 1$, we have $7R + I = R$, and one can check that 7 remains prime in R, S_2, S_3, and $\mathscr{O}_K$.

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Theorem (Coykendall-Kettinger-Liu-Mao-Zhao, 2025)

Let $R \subseteq T$ be domains, $p \in R$ prime, and $I := (R : T) \neq 0$ such that $pR + I = R$. Then p remains prime in T.

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Let $R \subseteq T$ be domains, $p \in R$ prime, and $I := (R : T) \neq 0$ such that $pR + I = R$. Then p remains prime in T and in every intermediate ring S with $R \subseteq S \subseteq T$.

- From the first theorem, we know that p remains prime in T.
- Let S satisfy $R \subseteq S \subseteq T$. Since the conductor grows when the rings get closer, we have $I = (R : T) \subseteq (R : S) =: I_S$.
- Thus, apply the first theorem for $R \subseteq S$, so that p remains prime in S.

Example (Vanishing Conductor)

Consider the extension

$$R = \mathbb{Z}[2x] \subset T = \mathbb{Z}[x].$$

where T is an integral overring of R. Here the conductor is

$$I = (R : T) = \{t \in T \mid tT \subseteq R\} = 0,$$

so the conductor vanishes. Let

$$p = 2x \in \mathbb{Z}[2x] = R.$$

In R, element p is prime because

$$\mathbb{Z}[2x]/(2x) \cong \mathbb{Z},$$

which is an integral domain. However, in $T = \mathbb{Z}[x]$, we can factor

$$2x = 2 \cdot x,$$

so p is no longer prime in T.

Prime Stability via Conductor Coprimality

Example

Let F be a field and set

$$R := F[x, y^2, y^3] \subset T := F[x, xy, y^2, y^3] \subset F[x, y].$$

Note that T is an integral overring of R (indeed $R \subset T \subset F[x, y]$).
Take $p := x$. The conductor $I := (R : T)$ is nonzero. In fact one can check precisely that

$$I = (R : T) = (y^2, y^3).$$

Notice that

$$R/(x, I) \cong R/(x, y^2, y^3) \cong F,$$

so (x, I) is a proper ideal of R. Hence $xR + I \neq R$, i.e. x and the conductor I are not coprime. In R, the element x is prime because

$$R/(x) \cong F[y^2, y^3]$$

is an integral domain. However in T we have the factorization

$$xy^3 = (xy) \cdot y^2,$$

and p divides neither factor. Hence, x is not prime in T.

Prime Stability via Conductor Coprimality

Definition

An order of an algebraic number field K is a subring $\mathscr{O} \subseteq \mathscr{O}_K$ which is also a $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$.

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Example

$\mathbb{Z}[i]$ is the ring of integers of number field $\mathbb{Q}(i)$, all orders are in the form $\mathbb{Z}[ni]$, where n is a positive integer.

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The following is a well-known theorem in order theory.

Theorem

Let K be a number field, $\mathscr{O}$ an order in K, and $\mathfrak{f} = (\mathscr{O} : \mathscr{O}_K)$ the conductor. A nonzero prime ideal $\mathfrak{p}$ of $\mathscr{O}$ is invertible if and only if $\mathfrak{p} + \mathfrak{f} = \mathscr{O}$.

Prime Stability via Conductor Coprimality

Corollary

Let $\mathscr{O}$ be an order in a number field K with conductor $\mathfrak{f} = (\mathscr{O} : \mathscr{O}_K)$. If $p \in \mathscr{O}$ is a prime element then p remains prime in $\mathscr{O}_K$ and in every intermediate order $\mathscr{O} \subseteq S \subseteq \mathscr{O}_K$.

Prime Stability via Conductor Coprimality

Corollary

Let $\mathcal{O}$ be an order in a number field K with conductor $\mathfrak{f} = (\mathcal{O} : \mathcal{O}_K)$. If $p \in \mathcal{O}$ is a prime element then p remains prime in $\mathcal{O}_K$ and in every intermediate order $\mathcal{O} \subseteq S \subseteq \mathcal{O}_K$.

- Any principal fractional ideal is invertible, hence $p\mathcal{O}$ is invertible.
- Since p is a prime element, $p\mathcal{O}$ is a prime ideal of $\mathcal{O}$.
- By the previous order theory theorem, $p\mathcal{O}$ is coprime to the conductor.
- Applying our conductor coprime theorem , p is prime in $\mathcal{O}_K$ and every intermediate order S.

Prime Stability via Conductor Coprimality

Corollary

Let $\mathscr{O}$ be an order in a number field K with conductor $\mathfrak{f} = (\mathscr{O} : \mathscr{O}_K)$. If $p \in \mathscr{O}$ is a prime element then p remains prime in $\mathscr{O}_K$ and in every intermediate order $\mathscr{O} \subseteq S \subseteq \mathscr{O}_K$.

- Recall a lemma by Dutta: If R is a Noetherian domain and T is the integral closure of R, then any prime element of R remains prime in T.
- $\mathscr{O}$ is an order (hence Noetherian), $\mathscr{O}_K$ is the integral closure of $\mathscr{O}$ (hence an integral overring), and both are 1-dimensional.
- Thus, any prime element $p \in \mathscr{O}$ remains prime in $\mathscr{O}_K$.
- Combining this with our 1-dimensional integral overring theorem shows that p remains prime in every intermediate order S.

Prime Stability via Conductor Coprimality

Corollary

Let $\mathscr{O}$ be an order in a number field K with conductor $\mathfrak{f} = (\mathscr{O} : \mathscr{O}_K)$. If $p \in \mathscr{O}$ is a prime element then p remains prime in $\mathscr{O}_K$ and in every intermediate order $\mathscr{O} \subseteq S \subseteq \mathscr{O}_K$.

The corollary also follows from a very classical order theory theorem.

Theorem

Let $\mathscr{O}$ be an order in K with conductor $\mathfrak{c}$. There exists a bijection between prime ideals in $\mathscr{O}_K$ and in $\mathscr{O}$ which are coprime to $\mathfrak{c}$.

Maximal Rings with Prime Stability

Definition

A partially ordered set is a set S with relation $\leq$ such that

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Example

The set of real numbers under the less than or equal to relation $\leq$ is a partially ordered set.

Example

Any set of sets under the inclusion relation $\subseteq$ is a partially ordered set.

Maximal Rings with Prime Stability

Definition

A chain in a partially ordered set S is a subset $C \subseteq S$ such that for any $a, b \in C$, we have $a \leq b$ or $b \leq a$.

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Note that $\{1, 2, 3\}$ is a chain in the partially ordered set of reals under the less than or equal to relation $\leq$ is a chain, as

$$1 \leq 2 \leq 3.$$

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Lemma (Zorn)

If a partially ordered set S has the property that every chain C has an upper bound, or an element $s \in S$ such that $s \geq c$ for all $c \in C$, then S contains a maximal element.

Maximal Rings with Prime Stability

Question (Coykendall-Kettinger-Liu-Mao-Zhao, 2025)

Given an extension $R \subseteq T$ and an element $p \in R$ prime in R, does there exist a ring $R \subseteq M \subseteq T$ which is maximal with respect to the property that p is prime in M?

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- Yes. Consider the partially ordered set of all intermediate rings for which p is prime, ordered under inclusion.
- Every chain has an upper bound, so by Zorn's lemma there exists a maximal element.

Maximal Rings with Prime Stability

Question (Coykendall-Kettinger-Liu-Mao-Zhao, 2025)

Given an extension $R \subseteq T$ and an element $p \in R$ *not* prime in R, does there exist a ring $R \subseteq M \subseteq T$ which is maximal with respect to the property that p is *not* prime in M?

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Given an extension $R \subseteq T$ and an element $p \in R$ *not* prime in R, does there exist a ring $R \subseteq M \subseteq T$ which is maximal with respect to the property that p is *not* prime in M?

- No. Consider the rings

$$A = \mathbb{Z}\left[y, y^{-1}\right] \text{ and } B = A\left[y^{-\frac{1}{2}}, y^{-\frac{1}{4}}, y^{-\frac{1}{8}}, \ldots\right].$$

We will let $R = A + xB[x]$ and $T = B[x]$.

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  We will let $R = A + xB[x]$ and $T = B[x]$.

- Using the identity $x^2 = \left(y^{\frac{1}{2^c}} x\right)\left(y^{-\frac{1}{2^c}} x\right)$, we can find that there is not a maximal intermediate ring where x is prime.

Definition

A chain of rings is a set C of rings such that for all $R, T \in C$, we have that $R \subseteq T$ or $T \subseteq R$.

Prime Stability in FCP Extensions

Definition

A chain of rings is a set C of rings such that for all $R, T \in C$, we have that $R \subseteq T$ or $T \subseteq R$.

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A ring extension T of R satisfies the Finite Chain Property (FCP) if every chain of intermediate rings is finite.

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Question

When does an element p of R prime in R and T remain prime in all intermediate rings?

Prime Stability in FCP Extensions

- Draw a directed graph on all intermediate rings under inclusion.

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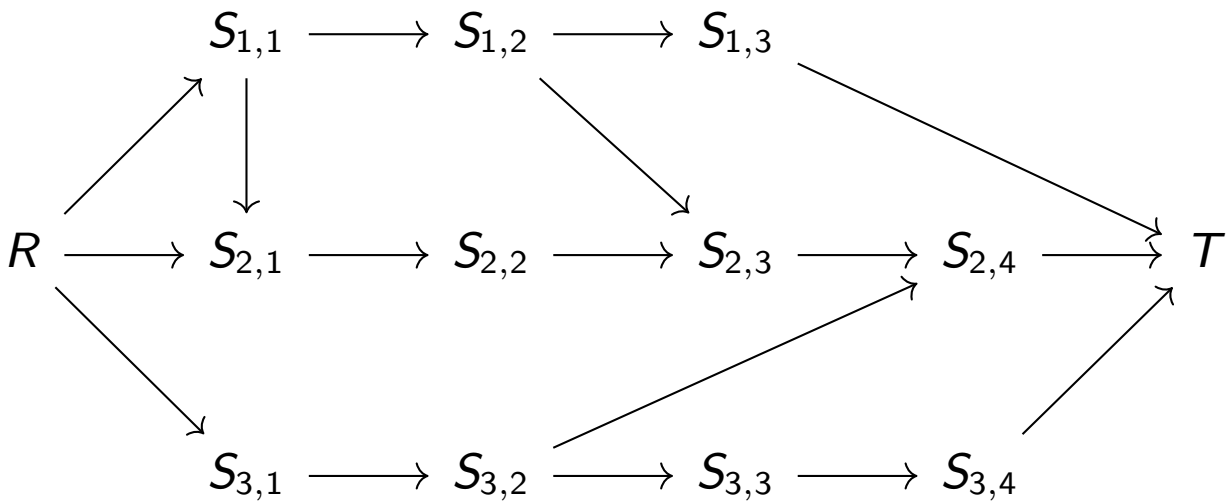

Figure: Ring extension $R \subset T$ satisfies FCP.

Definition

A ring extension $R \subsetneq T$ is minimal if there are no intermediate rings $S \neq R, T$. Equivalently, we say that R and T are adjacent.

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Theorem (Coykendall-Kettinger-Liu-Mao-Zhao, 2025)

Let $R \subset T$ be adjacent rings, and let p be a prime in R. Then p is either prime or a unit in T.

Prime Stability in FCP Extensions

Note that $T = R[a]$ for any $a \in T \setminus R$.

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Lemma

For a ring R with a prime p, suppose that $R \subseteq R[a] = R[pa]$, and that $r \in R$ satisfies $p \mid_R r$ if $p \mid_{R[a]} r$. Then p is prime in $R[a] = R[pa]$.

Note that $T = R[a]$ for any $a \in T \setminus R$.

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- Going back to the theorem, we consider

$$R \subseteq R[pa] \subseteq R[a] = T.$$

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- Going back to the theorem, we consider

$$R \subseteq R[pa] \subseteq R[a] = T.$$

- If p is not prime in T, it can be proved that $T = R\left[\frac{r}{p}\right]$ for $r \in R$.

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- If p is not prime in T, it can be proved that $T = R\left[\frac{r}{p}\right]$ for $r \in R$.
- We can then prove that p is a unit in T.

Prime Stability in FCP Extensions

Theorem (Coykendall-Kettinger-Liu-Mao-Zhao, 2025)

If T is a ring extension of R satisfying FCP and $p \in R$ is prime in both R and T, then p is prime in all intermediate rings.

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- Note that p only has to be a nonunit in T for this to hold.

Example

Consider the ring extension $\mathbb{Z}[ni] \subseteq \mathbb{Z}[i]$ for any positive integer n. This satisfies FCP, as all intermediate rings are of the form $\mathbb{Z}[di]$ for $d \mid n$, and so every chain has finite length. Therefore, if p is prime in $\mathbb{Z}[ni]$ and $\mathbb{Z}[i]$, then it is prime in every intermediate ring.

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- This also follows from the other results.

Example

Note that $\mathbb{Z} \subseteq \mathbb{Z}\left[\frac{1}{n}\right]$ also satisfies FCP for any positive integer n, as all intermediate rings are of the form $\mathbb{Z}\left[\frac{1}{d}\right]$ for $d \mid n$, and so every chain has finite length. Furthermore a prime p in both $\mathbb{Z}$ and $\mathbb{Z}\left[\frac{1}{n}\right]$ must satisfy $\gcd(p, n) = 1$, and therefore $\gcd(p, d) = 1$, so p is prime in $\mathbb{Z}\left[\frac{1}{d}\right]$. Therefore, this example satisfies the theorem.

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Example

For any positive integer n, define

$$R = \mathbb{Z} + x\mathbb{Q}[x], T = \mathbb{Z}\left[\frac{1}{n}\right] + x\mathbb{Q}[x].$$

All intermediate rings are then of the form $\mathbb{Z}\left[\frac{1}{d}\right] + x\mathbb{Q}[x]$ for some $d \mid n$. We see that T is an extension of R satisfying FCP, so every element prime in both R and T must be prime in every intermediate ring as well.

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- This example is not implied by the other results.
- Also, a prime $p \in R$ that is not a unit in $\mathbb{Q}[x]$ is prime in all $\mathbb{Z}\left[\frac{1}{n}\right] + x\mathbb{Q}[x]$.

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- This example is not implied by the other results.
- Also, a prime $p \in R$ that is not a unit in $\mathbb{Q}[x]$ is prime in all $\mathbb{Z}\left[\frac{1}{n}\right] + x\mathbb{Q}[x]$.
- Taking the union of the rings

$$R \subset \mathbb{Z}\left[\frac{1}{2}\right] + x\mathbb{Q}[x] \subset \mathbb{Z}\left[\frac{1}{2\cdot 3}\right] + x\mathbb{Q}[x] \subset \mathbb{Z}\left[\frac{1}{2\cdot 3\cdot 5}\right] + x\mathbb{Q}[x] \subset \cdots,$$

we find that if a prime in R is not a unit in $\mathbb{Q}[x]$, then it is a prime in $\mathbb{Q}[x]$.

Prime Stability in FCP Extensions

Theorem (Dobbs et al.)

If $R \subseteq S$ has conductor C, then $R \subseteq S$ is integral and satisfies FCP if and only if S is a finitely generated R-module such that R/C is an Artinian ring.

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Corollary

If $R \subseteq T$ has conductor C and T is a finitely generated R-module such that R/C is an Artinian ring, then if p is prime in R and not a unit in T, then p remains prime in every intermediate ring extension $R \subseteq S \subseteq T$.

Acknowledgements

We would like to deeply thank our mentors, Jared Kettinger and Prof. Jim Coykendall, for guiding us along our research, giving insightful feedback, and offering us encouragement. We would also like to thank the PRIMES program for making this experience possible.

References I

[1] Robert Gilmer and Jack Ohm. Integral domains with quotient overrings. *Mathematische Annalen*, 153(2):97–103, 1964.

[2] David E. Dobbs. On chains of overrings of an integral domain. *Trends in Commutative Rings Research*, page 173, 2004.

[3] Robert Gilmer. Some finiteness conditions on the set of overrings of an integral domain. *Proceedings of the American Mathematical Society*, 131(8):2337–2346, 2003.

[4] Paolo Zanardo. Intersections of powers of a principal ideal and primality. *Journal of Pure and Applied Algebra*, 188(1):287–304, 2004.

[5] Philip J. de Castro. Prime stability in intermediate extensions of integral domains. *Master's thesis*, Clemson University, 2020.

[6] David F. Anderson and David E. Dobbs. Pairs of rings with the same prime ideals. *Canadian Journal of Mathematics*, 32(2):362–384, 1980.

[7] David E. Dobbs, Gabriel Picavet, and Martine Picavet-L'Hermitte. Characterizing the ring extensions that satisfy FIP or FCP. *Journal of Algebra*, 371:391–429, 2012.

References II

[8] Frank-Olaf Schreyer. Integral ring extensions and Krull dimension. In *An Introduction to Algebraic Geometry: A Computational Approach*, pages 69–78. Springer, 2025.

[9] David E. Dobbs. On the commutative rings with at most two proper subrings. *International Journal of Mathematics & Mathematical Sciences*, 2016.

[10] J. C. Robson. Prime ideals in intermediate extensions. *Proceedings of the London Mathematical Society*, 3(2):372–384, 1982.

[11] David E. Dobbs. When the juxtaposition of two minimal ring extensions produces no new intermediate rings. *Palestine Journal of Mathematics*, 6(1):31–44, 2017.

[12] Amartya K. Dutta and Nobuharu Onoda. On finite generation of R-subalgebras of R[X]. *Journal of Algebra*, 320(1):57–80, 2008.

[13] David A. Cox. *Primes of the Form $x^2 + ny^2$: Fermat, Class Field Theory, and Complex Multiplication*. 3rd edition. American Mathematical Society, Providence, RI, 2022.

References III

[14] Keith Conrad. The conductor ideal of an order. Available at: https://kconrad.math.uconn.edu/blurbs/gradnumthy/conductor.pdf. Accessed August 24, 2025.